THE INDUCED DISJOINT PATHS PROBLEM ON (THETA, WHEEL)-FREE GRAPHS

Krstina Vušković University of Leeds, UK
Joint work with

Marko Radovanović University of Belgrade, Serbia

Nicolas Trotignon ENS de Lyon, France
G contains \( F \) if \( F \) is isomorphic to an induced subgraph of \( G \).

\( G \) is \textbf{\textit{F-free}} if it does not contain \( F \).

\( G \) is \textbf{\textit{F-free}} if \( G \) is \( F \)-free \( \forall f \in F \).

A \textbf{hole} in a graph is a chordless cycle of length \( \geq 4 \).
TRUEMPER CONFIGURATIONS

THETA

PYRAMID

PRISM

WHEEL

$\forall i \neq j \quad P_i \cup P_j$ induces a hole
PERFECT $\leq$ ODD-HOLE-FREE

\[ \arrow \]

NO PYRAMID, ODD WHEEL

\[ \text{Odd \# of sectors of length 1} \]

EVEN-HOLE-FREE

\[ \arrow \]

NO THETA, PRISM, EVEN WHEEL

\[ \text{Even \# of sectors} \]
<table>
<thead>
<tr>
<th>Property</th>
<th>Diagram 1</th>
<th>Diagram 2</th>
<th>Diagram 3</th>
<th>Diagram 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Universally Signable</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>CCKV ’96</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cap-Free</td>
<td>✓</td>
<td></td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>CCKV ’96</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Even-hole-Free</td>
<td>X</td>
<td>✓</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>CCKV ’97 DSV ’08</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Odd-hole-Free</td>
<td>✓</td>
<td>X</td>
<td>✓</td>
<td>SOME</td>
</tr>
<tr>
<td>Berge</td>
<td></td>
<td></td>
<td></td>
<td>SOME</td>
</tr>
<tr>
<td>Claw-Free</td>
<td>X</td>
<td>X</td>
<td>✓</td>
<td>SOME</td>
</tr>
<tr>
<td>Claw-Free CS ’07</td>
<td></td>
<td></td>
<td></td>
<td>SOME</td>
</tr>
<tr>
<td>Bull-Free</td>
<td>✓</td>
<td>X</td>
<td>✓</td>
<td>SOME</td>
</tr>
<tr>
<td>Bull-Free C ’10</td>
<td></td>
<td></td>
<td></td>
<td>SOME</td>
</tr>
<tr>
<td>ISK4-Free</td>
<td>✓</td>
<td>X</td>
<td>✓</td>
<td>SOME</td>
</tr>
<tr>
<td>ISK4-Free LMT ’13</td>
<td></td>
<td></td>
<td></td>
<td>SOME</td>
</tr>
<tr>
<td>Chordless Graphs</td>
<td>✓</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>MDTF ’13</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Only-Pyramid</td>
<td>X</td>
<td>✓</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Only-Pyramid DRTV</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Only-Prism</td>
<td>X</td>
<td>X</td>
<td>✓</td>
<td>X</td>
</tr>
<tr>
<td>Only-Prism DRTV</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Theta,Wheel)-Free</td>
<td>X</td>
<td>✓</td>
<td>✓</td>
<td>X</td>
</tr>
<tr>
<td>(Theta,Wheel)-Free RTV</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
RECOGNIZING TRUEMPER CONFIGURATIONS

(PYRAMID)

$O(n^9)$

SHORTEST PATHS DETECTOR TECHNIQUE

(THETA)

$O(n^n)$

$O(n^4)$ $\rightarrow$ $O(n^2 \log^2 n)$

(3-IN-A-TREE)

(PRISM)

NPC

(WHEEL)

NPC

(MAFFRAY, TROTIGNON '05)

(DIOT, TAVENAS, TROTIGNON '13)

(CHUDNOVSKY, SEYMOUR '05)

(CHUDNOVSKY, SEYMOUR '10)

(LAI, LU, THORUP '19)
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>RECOGNITION</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>$O(n^m)$</td>
</tr>
<tr>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>$O(n^7)$</td>
</tr>
<tr>
<td>X</td>
<td>X</td>
<td></td>
<td>X</td>
<td></td>
<td>$O(n^3m)$</td>
</tr>
<tr>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td>$O(n^7)$</td>
</tr>
<tr>
<td>X</td>
<td>X</td>
<td></td>
<td>X</td>
<td></td>
<td>$O(n^4m)$</td>
</tr>
<tr>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td>$O(n^3m)$</td>
</tr>
<tr>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td>$O(n^4m)$</td>
</tr>
<tr>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$O(n^7)$</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td>NPC</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>$O(n^5)$</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td>NPC</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td>$O(n^7)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td>NPC</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td>NPC</td>
</tr>
</tbody>
</table>
(RADOVANOVICH, TROTIGNON, VUŠKOVIC '17)

G \textbf{(Theta, Wheel)-free} ⇒

\textbf{for some clique} K

G \setminus K \textbf{ is}

\textbf{line graph of }Δ\textbf{-free}

\textbf{chordless graph}

\textbf{(every cycle is chordless)}

\textbf{clique cutset or 2-join}
BLOCKS OF DECOMPOSITION

G is \((\text{theta}, \text{wheel})\)-free \iff \forall i \ G_i \ is \ (\text{theta}, \text{wheel})\)-free

→ STRUCTURE THEOREM
DECOMPOSITION BASED RECOGNITION ALGORITHM

LEAVES: NO CLIQUE CUTSET, NO 2-JOIN

\( O(n^4m) \)
LOCAL STRUCTURE

$G$ (THETA,WHEEL)-FREE $\Rightarrow$

$\forall$ CLIQUE $K$ OF $G$

EITHER $K = V(G)$ OR $\exists$ BISIMPLICIAL VERTEX (OF $G$) IN $G \setminus K$

EASY ALGORITHM FOR $\omega$

$O(n^2m)$
(Trotignon, Vušković '09)

NO EXTREME 2-JOIN

NO STAR CUTSET $\Rightarrow$ EXTREME 2-JOIN

+ CAN DECOMPOSE BY SEQUENCE OF NON-CROSSING 2-JOINS
EXTREME 2-JOIN DECOMPOSITION TREE

NOTE: IF G IS (THETA, WHEEL)-FREE THEN
STAR CUTSET ⇒ ∃ CLIQUE CUTSET
SO NO CLIQUE CUTSET ⇒ NO STAR CUTSET

(TROTTIGNON, Vuškovic '09)
$\sigma(n^4 m)$
1 & 2-JOINS

GADGET

NOT CLASS-PRESERVING

G

ESSENTIAL TO DECOMPOSE WITH A SEQUENCE OF NON-CROSSING 2-JOINS

BASIC

EXTENDED BASIC
\[ a = \lambda_w(G[A_1 \cup C_1]) \]
\[ b = \lambda_w(G[B_1 \cup C_1]) \]
\[ c = \lambda_w(G[X_1]) \]
\[ d = \lambda_w(G[C_1]) \]

\[ \lambda_w(G^g_{2}) = \lambda(G) + d \]
Paths of length 33 whose interior vertices are of degree 2, and whose ends have no common neighbor

2-connected bipartite

Can decompose by sequence of non-crossing extreme 2-joins

Naves, Trotignon, Vušković '09

NPC for C
VERTEX COLORING

\[ G \text{ (THETA, WHEEL)-FREE} \Rightarrow \]

\[ G \text{ can be colored in } O(n^5m) \text{ time} \]

\[ \chi(G) \leq \max \{ 3, w(G) \} \]

s
2-IN-A-CYCLE

Decide whether $G$ contains a chordless cycle through 2 specified vertices $U$ and $V$

NPC in general (Bienstock '92)

$O(nm)$ for $(\theta, \text{WHEEL})$-free

$G$ $(\theta, \text{WHEEL})$-free, $U, V$ nonadjacent vertices

$\exists$ hole in $G$ that contains both $U$ and $V$

or clique cutset that separates $U$ and $V$

Note: Similar result holds for claw-free (Bruhn, Saito '12)
$k$-in-a-cycle

decide whether $G$ contains a chordless cycle through $k$ specified vertices $v_1, \ldots, v_k$

for $(\text{theta, wheel})$-free

fixed-parameter tractable

when parametrized by $k$
**Disjoint Paths (DP)**

\[ G, \mathcal{W} = \{(s_1, t_1), \ldots, (s_k, t_k)\} \]

Does there exist vertex-disjoint paths \( P_1, \ldots, P_k \) s.t. \( \forall i \ P_i = s_i \ldots t_i \)

- **\( k \) part of input \( \rightarrow \) NP-hard (Karp '75)**
- **\( k \) fixed (not part of input) \( \rightarrow \) \( k \)-DP \( O(n^3) \) (Robertson, Seymour '95)**

**Induced Disjoint Paths (IDP)**

\( P_1, \ldots, P_k \) vertex-disjoint + no edges between them

- \( k \)-IDP NPC whenever \( k \geq 2 \) (Bienstock '92)
\((\Theta, \text{wheel})\)-free: IDP NPC
\(r\)-IDP \(\mathcal{P} \bigcirc (n^{2k+6})\)

Claw-free: \(r\)-IDP \(\mathcal{P}\)
(Fiala, Kamiński, Lidický, Paulusma '12)
\(r\)-IDP FPT when parametrized by \(r\)
(Golovach, Paulusma, van Leeuwen '15)
FOR BASIC GRAPHS

IDP ESSENTIALLY REDUCES TO DP

ON ROOT GRAPH OF LINE GRAPH
CLIQUE CUTSETS

\[ G \subseteq \mathcal{G} \Rightarrow G \in \mathcal{G}_{\text{basic}} \text{ or CLIQUE CUTSET} \]

\[ O(n^c) \text{ k-IDP on } \mathcal{G}_{\text{basic}} \text{ (c constant that does not depend on } k \text{)} \]

\[ O(n^{2k+c}) \text{ k-IDP on } G \]
\((G, W)\) instance of \(k\)-1DP

\[ W = \{(s_i, t_i), \ldots, (s_k, t_k)\} \]

\[ W = \{s_1, \ldots, s_k, t_1, \ldots, t_k\} \text{ terminals of } W \]

In \(O(n^3)\) time we can find one of the following:

- K clique cutset

\[ |A \cap W| \geq 2 \]
\[ |B \cap W| \geq 2 \]

\(G'\): no clique cutset, so \(G' \subseteq G\) basic

A connected component \(C\) of \(G'\) is a clique \(N(C)\) is a clique

\( |C \cap W| \leq 1 \)
\[ W_A = \{ (s_i, t_i) : s_i, t_i \in A \} \]
\[ W_B = \{ (s_i, t_i) : s_i, t_i \in B \} \]
WE MAY ASSUME \( |W \setminus (W_A \cup W_B)| \leq 1 \)
(ELSE STOP WITH ANSWER NO)

CASE: \( |W \setminus (W_A \cup W_B)| = 0 \)

\[ \text{(G[A], W_A), (G[B], W_B)} \]

\[ \text{(G[A \cup \{x3\}, W_A), (G[B \setminus \{x\}], W_B))} \] \( \forall x \)

\[ \text{(G[A \cup \{x,y\}, W_A), (G[B \setminus (\{x\} \cup \{y\}], W_B))} \] \( \forall x,y \)

\[ \rightarrow O(n^2) \text{ PAIRS OF PROBLEMS S.T. (G,W) HAS SOLUTION IFF SOME PAIR OF INSTANCES HAS SOLUTION} \]
\((G, u)\) has a solution IFF \((G', u')\) has solution
2-joins

\((G, \mathcal{W})\) has a solution \iff in one of the cases below both \((G_1, \mathcal{W}_1)\) & \((G_2, \mathcal{W}_2)\) have solution

\[\mathcal{W} = \{ (s_i, t_i), (s_2, t_2), (s_3, t_3) \}\]

\[\mathcal{W}_1 = \{ (s_i, t_i), (s_3, s_2), (b_2, b_2') \}\]

\[\mathcal{W}_2 = \{ (s_2, t_2), (a_1, t_3), (b_1, d_1) \}\]
\(G_{F,\sigma} = (G, F, \sigma)\)

\[G = \text{GRAPH} \]

\[F = \text{SET OF SOME FLAT PATHS OF LENGTH } \leq 7 + \text{SOME PATHS OF LENGTH 0} \]

\[\sigma = \text{SET OF TERMINAL PAIRS S.T.} \]

\[\forall \omega \in \sigma \ (G, \omega) \text{ IS AN INSTANCE OF 1DP WHERE EVERY TERMINAL IS CONTAINED IN A PATH } P \in F \]

\(G_{F,\sigma} \text{ IS LINKABLE IF FOR AT LEAST ONE } \omega \in \sigma, (G, \omega) \text{ HAS SOLUTION} \)

\(G_{F,\sigma} \sigma\text{-GRAPH S.T. } |F| \leq t \Rightarrow |\sigma| \leq 2^{8t} (8t)! \)

\(G_{F,\sigma} \sigma\text{-GRAPH S.T. } G \text{ BASIC} & |F| \leq t \Rightarrow \text{CAN DECIDE IN } O(n^5) \text{ TIME WHETHER } G_{F,\sigma} \text{ IS LINKABLE} \)
Solve a bunch of problems on $G'$, record potential solutions in $G_{F_2, \sigma_2}$.

$G_{F, \sigma}$ linkable IFF $G^2_{F_2, \sigma_2}$ linkable.