Did Erik Palmgren Solve a Revised Hilbert’s Program?

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In Living Memory of Erik Palmgren
Erik Palmgren, 1963 - 2019

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Introduction to Martin-Löf Type Theory

Interpretation of Iterated Inductive Definitions
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Martin-Löf Type Theory

- **Martin-Löf Type Theory** (MLTT) is a type theory for formalising constructive mathematics.
- It is designed in such a way that one has – as far as possible – a direct insight into the validity of its judgements.
  - As a response to the failure of the original Hilbert’s program due to Gödel’s 2nd Incompleteness Theorem.
- MLTT is as well the basis for the theoretical basis for the interactive theorem prover and dependently typed programming language *Agda.*
Dependent Type Theory

- Simple Type Theory has non dependent types, the main ones being
  \[ A \times B \quad A \rightarrow B \]

- Dependent Type Theory allows types to dependent on elements of other types.

- One of the origins is the interpretation of the \( \forall \)-quantifier.
  - In BHK interpretation of logical connectives, a
    **proof of** \( \forall x : A. B \; x \)
    - is a **function** that
      - maps an element \( x : A \) to a proof of \( B \; x \).
  - So proofs are elements of type \( \Pi A \; B \).
  - \( \Pi A \; B \) = type of dependent functions, which map
    \( x : A \) to an element of \( B \; x \).

- **Remark:** \( \text{Set} \) in MLTT is what is usually called “Type”. 
Π-Type

- **Formation rule:**

\[
\begin{array}{c}
A : \text{Set} \\
B : A \to \text{Set}
\end{array}
\quad \quad \quad
\begin{array}{c}
\Pi A B : \text{Set}
\end{array}
\]

- **Introduction rule:**

\[
x : A \Rightarrow t : B x \\
\lambda x.t : \Pi A B
\]

- **Elimination rule:**

\[
f : \Pi A B \\
a : A
\quad \quad \quad
\begin{array}{c}
\text{Ap} f a : B a
\end{array}
\]

- **Equality rule:**

\[
x : A \Rightarrow t : B x \\
a : A
\quad \quad \quad
\begin{array}{c}
\text{Ap} (\lambda x.t) a = t[x := a] : B a
\end{array}
\]
**W-Type**

**Assume** $A : \text{Set}, \ B : A \to \text{Set}$.

$W_{A,B}$ is the type of well-founded recursive trees with branching degrees $(B \ a)_{a : A}$.

If $B \ a''$ empty we get a leaf $f' \ z = \sup_{a''} f''$

$$
\begin{align*}
& f' \ z' \\
& z' \\
& z : B \ a'
\end{align*}
$$

$$
\begin{align*}
& f \ y' \\
& y' \\
& y : B \ a
\end{align*}
$$

$$
\begin{align*}
& \sup_{a} f \\
& (a : A, f : B \ a \to W_{A,B})
\end{align*}
$$
Kleene’s O

Example Kleene’s O, tree of height $\omega$, Version in MLTT.

$$\text{KleeneO}_{\text{ML}} := W \ A \ B,$$

where $A = \{\hat{\emptyset}, \hat{*}, \hat{\mathbb{N}}\}$

$B \hat{\emptyset} = \emptyset \quad B \hat{*} = \{\ast\} \quad B \hat{\mathbb{N}} = \mathbb{N}.$
Example Kleene O₂:

\[
\text{KleeneO}_{ML,2} := W \ A' \ B' \text{ where }
\]

\[
\begin{align*}
A' &= \{\hat{\emptyset}, \hat{*}, \hat{\mathbb{N}}, \text{KleeneO}\} \\
B' : A' &\to \text{Set} \\
B' \hat{\emptyset} &= \emptyset \\
B' \hat{*} &= \{*\} \\
B' \hat{\mathbb{N}} &= \mathbb{N} \\
B' \hat{\text{KleeneO}} &= \text{KleeneO}_{ML}
\end{align*}
\]

Therefore it’s a \textit{nested W-type}.

We can define \(\omega^\text{ck}_1 : \text{KleeneO}_{ML,2}\),

\[
\omega^\text{ck}_1 := \sup \text{KleeneO embed}
\]

\(\text{embed} : \text{KleeneO}_{ML} \to \text{KleeneO}_{ML,2}\) embedding function.

\(\omega^\text{ck}_1\) has height the supremum of the heights of all elements in \(\text{KleeneO}_{ML}\).
The \( W \)-Type

- **Formation rule:**
  \[
  \begin{align*}
  A : \text{Set} & \quad B : A \to \text{Set} \\
  \hline
  W A B : \text{Set}
  \end{align*}
  \]

- **Introduction rule:**
  \[
  \begin{align*}
  a : A & \quad b : B \to W A B \\
  \hline
  \sup a b : W A B
  \end{align*}
  \]

- **Elimination and Equality Rules:** Induction over trees.
Universes

- A universe is a family of sets
- Given by
  - an set $U : \text{Set}$ of \textit{codes} for sets,
  - a \textit{decoding function} $T : U \to \text{Set}$.
- **Formation rules:**
  $$U : \text{Set} \quad T : U \to \text{Set}$$
- **Introduction and Equality rules:**
  $$\hat{N} : U \quad T \hat{N} = N$$
  
  $$a : U \quad b : T a \to U$$
  $$\hat{\Pi} a b : U$$
  (compare with $A : \text{Set} \quad b : A \to \text{Set}$)

  $$T(\hat{\Pi} a b) = \Pi (T a)(T \circ b)$$

  Similarly for other type formers (except for $U$).
Introduction to Martin-Löf Type Theory

Interpretation of Iterated Inductive Definitions
ID^i is the theory of intuitionistic inductive definitions given by

- The language and theory HA of Heyting Arithmetic,
- for formulas \( \mathcal{A}(X, y) \) strictly positive in \( X \)
  - a predicate \( I_{\mathcal{A}} \) (written \( n \in I_{\mathcal{A}} \))
  - axioms expressing that \( I_{\mathcal{A}} \) is the least set closed under \( \mathcal{A} \):

\[
\forall n. \mathcal{A}(I_{\mathcal{A}}, n) \rightarrow n \in I_{\mathcal{A}}
\]

\[
\forall n \in I_{\mathcal{A}}. \mathcal{A}(B, n) \rightarrow B(n)
\]

\[
\forall n \in I_{\mathcal{A}}. B(n)
\]

where \( B(x) \) is any formula with distinguished variable \( x \), which might make use of \( I_{\mathcal{A}} \).
**Example: Inductive Definition of Kleene’s O**

- **KleeneO** (Kleene’s O as a set of natural numbers) can be defined **inductively** by
  - $\langle 0, 0 \rangle \in \text{KleeneO}.$
  - If $e \in \text{KleeneO}$ then $\langle 1, e \rangle \in \text{KleeneO}$
  - If $\forall n \in \mathbb{N}. \{e\}(n) \in \text{KleeneO},$ then $\langle 2, e \rangle \in \text{KleeneO}.$

- Definable in $\text{ID}^i$ using
  $$A(X, n) :=$$
  $$( n = \langle 0, 0 \rangle$$
  $$\lor (\exists m. n = \langle 1, m \rangle \land m \in X)$$
  $$\lor (\exists e. n = \langle 1, e \rangle \land \forall m. \exists k. \{e\}(m) \approx k \land k \in X))$$

- So the above definition is equivalent to the **inductive definition**

  if $A(\text{KleeneO}, n)$ then $n \in \text{KleeneO}$
Kleene’s $O$ as subset of $\mathbb{N}$
ID\textsuperscript{i} is the smallest (in a proof theoretic sense) fully impredicative theory studied in proof theory.\textsuperscript{2}

It’s strength is the Bachmann Howard Ordinal, in modern notation (e.g. [5])

$$\psi_{\Omega_1}(\epsilon_{\Omega_1+1})$$

Iterated inductive definitions were the topic of the famous monograph “BuFePoSi” [2].

\textsuperscript{2}There is another notion of predicativity which gives limit $\Gamma_0$. Jäger calls theories between $\Gamma_0$ and ID\textsuperscript{i} “meta-predicative”.
Theory of Finitely Iterated Intuitionistic Inductive Definitions

- $\text{ID}_n^i$ is the theory of $n$ times iterated inductive definition.
- Allows predicates $I_{A,k}$ for $k < n$
  where $I_{A,k}$ can refer to $I_{A',k'}$ for $k' < k$ (positively and negatively).
  - Kleene$O_2$ can be defined in $\text{ID}_2^i$ as one inductive definition which
    refers to Kleene$O$.
  - Can be generalised to Kleene$O_n$, definable in $\text{ID}_n^i$.
- $\text{ID}_n^i = \psi_{\Omega_1}(\epsilon_{\Omega_{n+1}})$ (e.g. [5]).
- $\text{ID}_{<\omega}^i$ is the union of $\text{ID}_n^i$ and has strength $\psi_{\Omega_1}(\Omega_\omega) = |(\Pi^1_1 - \text{CA})_0|$. 
We define the theory $\mathbf{ID}_\alpha^i$ of transfinitely iterated intuitionistic inductive definitions:

Fix an ordinal notation system $(\mathcal{O}_T, \prec)$ of order type $\alpha$, i.e.

- $\mathcal{O}_T \subseteq \mathbb{N}$ primitive recursive,
- $\prec$ primitive recursive binary relation on $\mathcal{O}_T$,
- $(\mathcal{O}_T, \prec)$ well founded of order type $\alpha$.
- $\beta, \gamma, \ldots$ refer to elements of $\mathcal{O}_T$.

Language of $\mathbf{ID}_\alpha^i$ is given by

- for any predicate $\mathcal{A}(X, Y, \beta, n)$ strictly positive in $X$
  - a binary predicate symbol $n \in \mathbf{I}_{\mathcal{A}, \beta}$
  - a defined predicate

$$\mathbf{I}_{\mathcal{A}, \prec \beta} := \bigcup_{\gamma \prec \beta} \{\gamma\} \times \mathbf{I}_{\mathcal{A}, \gamma}$$
Theory of transfinitely iterated intuitionistic inductive definitions

▶ Axioms

\[
\begin{align*}
\beta \in \text{OT} & \quad \mathcal{A}(\mathcal{I}_A, \beta, \mathcal{I}_A, \prec \beta, \beta, n) \\
\Rightarrow & \quad n \in \mathcal{I}_A, \beta
\end{align*}
\]

\[
\begin{align*}
\beta \in \text{OT} & \quad \forall n \in \mathcal{I}_A, \beta. \mathcal{A}(B, \mathcal{I}_A, \prec \beta, \beta, n) \rightarrow B(n) \\
\Rightarrow & \quad \forall n \in \mathcal{I}_A, \beta. B(n)
\end{align*}
\]

▶ Transfinite induction over OT.

▶ $\text{ID}^i_{\prec \alpha}$ is the union of the theories $\text{ID}^i_{\beta}$ for $\beta < \alpha$. 
Eric Palmgren was able to interpret $\text{ID}^i_{<\epsilon_0}$ in

$$\text{ML}_1^W := \text{MLTT} + W + U$$

This showed that the proof theoretic strength of the type theory in question is

$$|\text{ML}_1^W| \geq |\text{ID}^i_{<\epsilon_0}| = |\Delta^1_2 - \text{CA}| = \psi_{\Omega_1}(\Omega_{\epsilon_0})$$

In our PhD thesis [6, 7] we showed that the strength is much bigger

$$|\text{ML}_1^W| = \psi_{\Omega_1}(\Omega_{I+\omega})$$

The proof required advanced well-ordering techniques due to Buchholz and Pohlers.\(^3\)

\(^3\)Jäger might have been involved as well - I haven't investigated that yet. Our approach was based on the refined version by Buchholz, in draft version [1], see as well the book by Buchholz and Schütte [3]
Palmgren’s Results as a Solution to a revised Hilbert’s Program

- By Palmgren’s result, the strength of MLTT with $W$-type and one universe is $> |(\Pi_1^1 - CA)_0|$, which is the biggest of the big 5 systems in reverse mathematics [9].
- $(\Pi_1^1 - CA)_0$ allows to prove therefore most “real” mathematical theories.
- $ML_1 W$ proves its consistency.
- $ML_1 W$ was designed to be a trustworthy theory (meaning explanations). ⁴
- If one trusts in this type theory, one can trust in the correctness of those proofs.
- Therefore Palmgren’s result gives a first quite strong solution to a revised Hilbert’s program.

⁴Trustworthiness is subject to a philosophical debate
When revisiting Palmgren’s proof one sees that he didn’t use the full power of \( ML_{1W} \).
- We can restrict \( W \)-types to elements of the universe.
  So we define \( (W \ a \ b) \) only for \( a : U \) and \( b : T \ a \to U \).
- We can restrict induction over \( W \)-types to elements of the universe.
- Let the resulting theory be called \( ML_{1W^-} \).

Subject to working out the full details of the proof we obtain the following result [8]:
- The interpretation of \( ID_{<\epsilon_0}^i \) by Palmgren can be carried out as well in \( ML_{1W^-} \).
- \( ML_{1W^-} \) can be interpreted in \( ID_{<\epsilon_0}^i \)
- Therefore \( |ML_{1W^-}| = |ID_{<\epsilon_0}^i| = \psi_{\Omega_1}(\Omega_{\epsilon_0}) \).
Palmgren showed that $\text{ID}^i_{<\epsilon_0}$ can be interpreted in $\text{ML}_1 W$.

Therefore $\text{ML}_1 W$ shows the consistency of $(\Pi^1_1 - \text{CA})_0$ sufficient to carry out most real mathematical proofs.

Therefore Palmgren’s result gives an answer to a revised Hilbert’s program.

The result can be sharpened to determine the precise strength of a weaker theory $\text{ML}_1 W^-$.
Strictly positive inductive definitions give rise to a monotone operator

\[ \Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) \]

where \( \mathcal{P}(\mathbb{N}) = \mathbb{N} \to U \).

For a strictly positive inductive definition one can “collect” all the sets, \( \forall \)-quantifiers in its definition are ranging over.

Now one defines a \( W \)-type which has as branching degree all those sets.

If we iterate the operator \( \Gamma \) transfinitely over the \( W \)-type, one obtains the least fixed point of \( \Gamma \) which one can use to interpret an inductive definition.

By “Gentzen’s trick” one obtains transfinite induction up to \( < \varepsilon_0 \) over types, and can use it to get iterated inductive definitions up to \( \alpha \) for any \( \alpha < \varepsilon_0 \).
Erik Palmgren showed as well in [4] that if one replaces the $W$-type in type theory by finitely iterated versions of Aczel’s $V$ used by Aczel to interpret constructive set theory CZF in type theory one obtains the strength $|\text{ID}^i_{\omega}| = \psi_{\Omega_1}(\Omega_\omega) = |(\Pi^1_1 - CA)_0|$ (as noted before)
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