Did Erik Palmgren Solve a Revised Hilbert's Program?

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In Living Memory of Erik Palmgren

Erik Palmgren, 1963 - 2019

1

[1] Source:
https://www.math.su.se/om-oss/nyheter/erik-palmgren-1963-2019-1.463835

Introduction to Martin-Löf Type Theory

Interpretation of Iterated Inductive Definitions

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Interpretation of Iterated Inductive Definitions

Martin-Löf Type Theory

- **Martin-Löf Type Theory** (**MLTT**) is a type theory for formalising constructive mathematics.
- It is designed in such a way that one has – as far as possible – a **direct insight into the validity of its judgements**.
  - As a response to the failure of the original Hilbert's program due to Gödel's 2nd Incompleteness Theorem.
- MLTT is as well the basis for the theoretical basis for the interactive theorem prover and dependently typed programming language **Agda**.

Dependent Type Theory

- Simple Type Theory has non dependent types, the main ones being

$$A \times B \qquad A \to B$$

- Dependent Type Theory allows types to dependent on elements of other types.
- One of the origins is the interpretation of the $\forall$-quantifier.
  - In BHK interpretation of logical connectives, a **proof of $\forall x : A.B\ x$**
    - is a **function** that
    - maps an element $x : A$ to a proof of $B\ x$.
  - So proofs are elements of type $\Pi\ A\ B$.
  - $\Pi\ A\ B$ = type of dependent functions, which map $x : A$ to an element of $B\ x$.
- **Remark:** Set in MLTT is what is usually called "Type".

Π-Type

- **Formation rule:**

$$\frac{A : \mathrm{Set} \qquad B : A \to \mathrm{Set}}{\Pi\ A\ B : \mathrm{Set}}$$

- **Introduction rule:**

$$\frac{x : A \Rightarrow t : B\ x}{\lambda x.t : \Pi\ A\ B}$$

- **Elimination rule:**

$$\frac{f : \Pi\ A\ B \qquad a : A}{\mathrm{Ap}\ f\ a : B\ a}$$

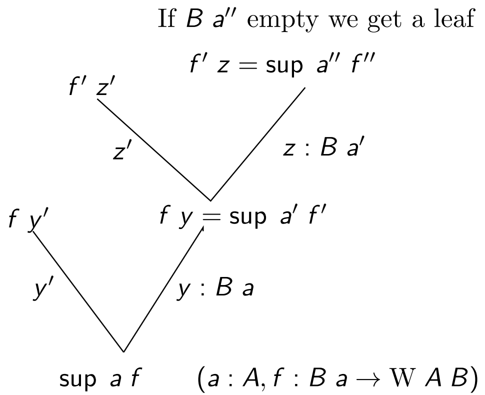
- **Equality rule:**

$$\frac{x : A \Rightarrow t : B\ x \qquad a : A}{\mathrm{Ap}\ (\lambda x.t)\ a = t[x := a] : B\ a}$$

W-Type

Assume $A : \mathrm{Set}$, $B : A \to \mathrm{Set}$.

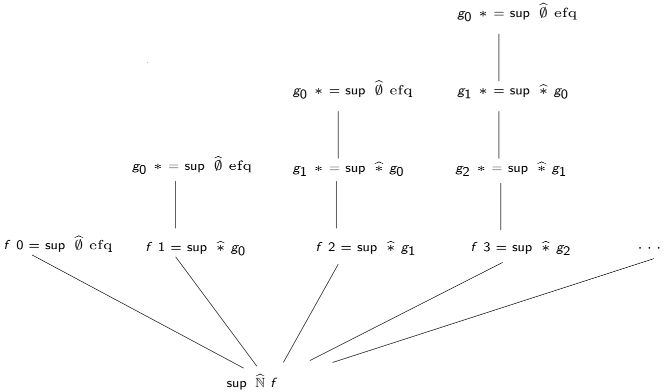
$\mathrm{W}\ A\ B$ is the type of well-founded recursive trees with branching degrees $(B\ a)_{a:A}$.



Kleene's O

Example Kleene's O, tree of height ω, Version in MLTT.
$\mathrm{KleeneO_{ML}} := \mathrm{W}\ A\ B$, where $A = \{\widehat{\emptyset}, \widehat{*}, \widehat{\mathbb{N}}\}$

$B\ \widehat{\emptyset} = \emptyset \qquad B\ \widehat{*} = \{*\} \qquad B\ \widehat{\mathbb{N}} = \mathbb{N}.$



$\mathrm{KleeneO}_{\mathrm{ML},2}$

Example Kleene O_2:

- $\mathrm{KleeneO}_{\mathrm{ML},2} := \mathrm{W}\ A'\ B'$ where
  - $A' = \{\widehat{\emptyset}, \widehat{*}, \widehat{\mathbb{N}}, \widehat{\mathrm{KleeneO}}\}$
  - $B' : A' \to \mathrm{Set}$

    $$\begin{array}{lcl} B'\ \widehat{\emptyset} & = & \emptyset \\ B'\ \widehat{*} & = & \{*\} \\ B'\ \widehat{\mathbb{N}} & = & \mathbb{N} \\ B'\ \widehat{\mathrm{KleeneO}} & = & \mathrm{KleeneO}_{\mathrm{ML}} \end{array}$$
  - Therefore it's a **nested W-type**.
- We can define $\omega_1^{\mathrm{ck}} : \mathrm{KleeneO}_{\mathrm{ML},2}$,

  $\omega_1^{\mathrm{ck}} := \mathsf{sup}\ \widehat{\mathrm{KleeneO}}\ \mathrm{embed}$

  $\mathrm{embed} : \mathrm{KleeneO}_{\mathrm{ML}} \to \mathrm{KleeneO}_{\mathrm{ML},2}$ embedding function.
- ω_1^{ck} has height the supremum of the heights of all elements in $\mathrm{KleeneO}_{\mathrm{ML}}$.

The W-Type

- **Formation rule:**

$$\frac{A : \mathrm{Set} \qquad B : A \to \mathrm{Set}}{\mathrm{W}\ A\ B : \mathrm{Set}}$$

- **Introduction rule:**

$$\frac{a : A \qquad b : B\ a \to \mathrm{W}\ A\ B}{\mathsf{sup}\ \ a\ b : \mathrm{W}\ A\ B}$$

- **Elimination and Equality Rules:** Induction over trees.

Universes

- A universe is a family of sets
- Given by
  - an set $\mathrm{U} : \mathrm{Set}$ of **codes** for sets,
  - a **decoding function** $\mathrm{T} : \mathrm{U} \to \mathrm{Set}$.
- **Formation rules:**

$$\mathrm{U} : \mathrm{Set} \qquad \mathrm{T} : \mathrm{U} \to \mathrm{Set}$$

- **Introduction and Equality rules:**

$$\widehat{\mathbb{N}} : \mathrm{U} \qquad \mathrm{T}\ \widehat{\mathbb{N}} = \mathbb{N}$$

$$\frac{a : \mathrm{U} \qquad b : \mathrm{T}\ a \to \mathrm{U}}{\widehat{\Pi}\ a\ b : \mathrm{U}}$$ (compare with $\frac{A : \mathrm{Set} \qquad b : A \to \mathrm{Set}}{\Pi\ A\ B : \mathrm{Set}}$)

$$\mathrm{T}(\widehat{\Pi}\ a\ b) = \Pi\ (\mathrm{T}\ a)\ (\mathrm{T} \circ b)$$

Similarly for other type formers (except for U).

Introduction to Martin-Löf Type Theory

Interpretation of Iterated Inductive Definitions

Theory of Intuitionistic Inductive Definitions

- ID^{i} is the theory of intuitionistic inductive definitions given by
  - The language and theory HA of Heyting Arithmetic,
  - for formulas $\mathcal{A}(X, y)$ strictly positive in X
    - a predicate $\mathrm{I}_{\mathcal{A}}$ (written $n \in \mathrm{I}_{\mathcal{A}}$)
    - axioms expressing that $\mathrm{I}_{\mathcal{A}}$ is the least set closed under $\mathcal{A}$:

$$\forall n.\mathcal{A}(\mathrm{I}_{\mathcal{A}}, n) \to n \in \mathrm{I}_{\mathcal{A}}$$

$$\frac{\forall n \in \mathrm{I}_{\mathcal{A}}.\mathcal{A}(B, n) \to B(n)}{\forall n \in \mathrm{I}_{\mathcal{A}}.B(n)}$$

where $B(x)$ is any formula with distinguished variable x, which might make use of $\mathrm{I}_{\mathcal{A}}$.

Example: Inductive Definition of Kleene's O

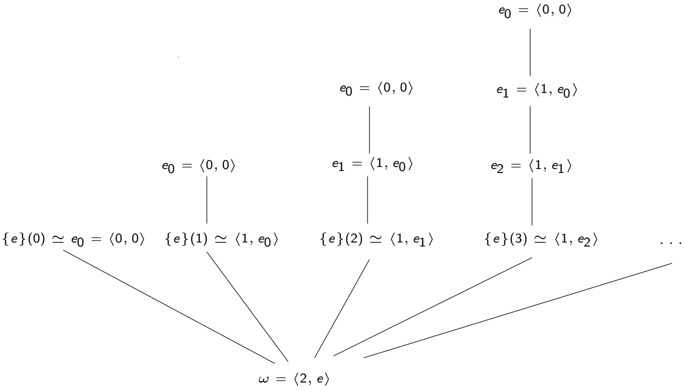
- $\mathrm{KleeneO}$ (Kleene's O as a set of natural numbers) can be defined **inductively** by
  - $\langle 0, 0 \rangle \in \mathrm{KleeneO}$.
  - If $e \in \mathrm{KleeneO}$ then $\langle 1, e \rangle \in \mathrm{KleeneO}$
  - If $\forall n \in \mathbb{N}.\{e\}(n) \in \mathrm{KleeneO}$, then $\langle 2, e \rangle \in \mathrm{KleeneO}$.
- Definable in ID^{i} using
  $\mathcal{A}(X, n) :=$

$$
\begin{array}{l}
(n = \langle 0, 0 \rangle \\
\ \vee (\exists m. n = \langle 1, m \rangle \wedge m \in X) \\
\ \vee (\exists e. n = \langle 1, e \rangle \wedge \forall m. \exists k. \{e\}(m) \simeq k \wedge k \in X))
\end{array}
$$

- So the above definition is equivalent to the **inductive definition**

$$
\text{if} \quad \mathcal{A}(\mathrm{KleeneO}, n) \quad \text{then} \quad n \in \mathrm{KleeneO}
$$

Kleene's O as subset of $\mathbb{N}$



Theory of Intuitionistic Inductive Definitions

- ID^{i} is the smallest (in a proof theoretic sense) fully impredicative theory studied in proof theory.[2]
- It's strength is the Bachmann Howard Ordinal, in modern notation (e.g. [5])

$$\psi_{\Omega_1}(\epsilon_{\Omega_1+1})$$

- Iterated inductive definitions were the topic of the famous monograph "BuFePoSi" [2].

[2] There is another notion of predicativity which gives limit Γ_0.
Jäger calls theories between Γ_0 and $\mathrm{ID}^{\mathrm{i}}_1$ "meta-predicative".

Theory of Finitely Iterated Intuitionistic Inductive Definitions

- $\mathrm{ID}^{\mathrm{i}}_{n}$ is the theory of n times iterated inductive definition.
- Allows predicates $\mathrm{I}_{\mathcal{A},k}$ for $k < n$ where $\mathrm{I}_{\mathcal{A},k}$ can refer to $\mathrm{I}_{\mathcal{A}',k'}$ for $k' < k$ (positively and negatively).
  - $\mathrm{KleeneO}_2$ can be defined in $\mathrm{ID}^{\mathrm{i}}_{2}$ as one inductive definition which refers to $\mathrm{KleeneO}$.
  - Can be generalised to $\mathrm{KleeneO}_n$, definable in $\mathrm{ID}^{\mathrm{i}}_{n}$.
- $\mathrm{ID}^{\mathrm{i}}_{n} = \psi_{\Omega_1}(\epsilon_{\Omega_n+1})$ (e.g. [5]).
- $\mathrm{ID}^{\mathrm{i}}_{<\omega}$ is the union of $\mathrm{ID}^{\mathrm{i}}_{n}$ and has strength $\psi_{\Omega_1}(\Omega_\omega) = |(\Pi^1_1 - \mathrm{CA})_0|$.

Theory of transfinitely iterated intuitionistic inductive definitions

- We define the theory $\mathrm{ID}^{\mathrm{i}}_{\alpha}$ of transfinitely iterated intuitionistic inductive definitions:
- Fix an ordinal notation system $(\mathrm{OT}, \prec)$ of order type α, i.e.
  - $\mathrm{OT} \subseteq \mathbb{N}$ primitive recursive,
  - $\prec$ primitive recursive binary relation on OT,
  - $(\mathrm{OT}, \prec)$ well founded of order type α.
  - $\beta, \gamma, \ldots$ refer to elements of OT.
- Language of $\mathrm{ID}^{\mathrm{i}}_{\alpha}$ is given by
  - for any predicate $\mathcal{A}(X, Y, \beta, n)$ strictly positive in X
    - a binary predicate symbol $n \in \mathrm{I}_{\mathcal{A},\beta}$
    - a defined predicate

$$\mathrm{I}_{\mathcal{A},\prec\beta} := \bigcup_{\gamma\prec\beta} \{\gamma\} \times \mathrm{I}_{\mathcal{A},\gamma}$$

Theory of transfinitely iterated intuitionistic inductive definitions

- Axioms

$$\frac{\beta \in \mathrm{OT} \qquad \mathcal{A}(\mathrm{I}_{\mathcal{A},\beta}, \mathrm{I}_{\mathcal{A},\prec\beta}, \beta, n)}{n \in \mathrm{I}_{\mathcal{A},\beta}}$$

$$\frac{\beta \in \mathrm{OT} \qquad \forall n \in \mathrm{I}_{\mathcal{A},\beta}.\mathcal{A}(B, \mathrm{I}_{\mathcal{A},\prec\beta}, \beta, n) \to B(n)}{\forall n \in \mathrm{I}_{\mathcal{A},\beta}.B(n)}$$

- Transfinite induction over OT.
- $\mathrm{ID}^{\mathrm{i}}_{\prec\alpha}$ is the union of the theories $\mathrm{ID}^{\mathrm{i}}_{\beta}$ for $\beta \prec \alpha$.

Interpretation of Palmgren [4]

- Eric Palmgren was able to interpret $\mathrm{ID}^{\mathrm{i}}_{<\epsilon_0}$ in

$$\mathrm{ML}_1\mathrm{W} := \mathsf{MLTT} + \mathrm{W} + \mathrm{U}$$

  - This showed that the proof theoretic strength of the type theory in question is

$$|\mathrm{ML}_1\mathrm{W}| \geq |\mathrm{ID}^{\mathrm{i}}_{<\epsilon_0}| = |\Delta^1_2 - \mathrm{CA}| = \psi_{\Omega_1}(\Omega_{\epsilon_0})$$

  - In our PhD thesis [6, 7] we showed that the strength is much bigger

$$|\mathrm{ML}_1\mathrm{W}| = \psi_{\Omega_1}(\Omega_{\mathrm{I}+\omega})$$

  - The proof required advanced well-ordering techniques due to Buchholz and Pohlers.[3]

[3] Jäger might have been involved as well - I haven't investigated that yet.
Our approach was based on the refined version by Buchholz, in draft version [1], see as well the book by Buchholz and Schütte [3]

Palmgren's Results as a Solution to a revised Hilbert's Program

- By Palmgren's result, the strength of MLTT with W-type and one universe is $> |(\Pi^1_1 - \mathrm{CA})_0|$, which is the biggest of the big 5 systems in reverse mathematics [9].
- $(\Pi^1_1 - \mathrm{CA})_0$ allows to prove therefore most "real" mathematical theories.
- $\mathrm{ML}_1\mathrm{W}$ proves its **consistency**.
- $\mathrm{ML}_1\mathrm{W}$ was designed to be a **trustworthy** theory (meaning explanations).[4]
- If one trusts in this type theory, one can trust in the correctness of those proofs.
- Therefore Palmgren's result gives a a first quite strong solution to a **revised Hilbert's program**.

[4] Trustworthiness is subject to a philosophical debate

Sharpening the Bounds of Palmgren

- When revisiting Palmgren's proof one sees that he didn't use the full power of $\mathrm{ML}_1\mathrm{W}$.
  - We can restrict W-types to elements of the universe. So we define $(\mathrm{W}\ a\ b)$ only for $a : \mathrm{U}$ and $b : \mathrm{T}\ a \to \mathrm{U}$.
  - We can restrict induction over W-types to elements of the universe.
  - Let the resulting theory be called $\mathrm{ML}_1\mathrm{W}^-$.
- Subject to working out the full details of the proof we obtain the following result [8]:
  - The interpretation of $\mathrm{ID}^{\mathrm{i}}_{<\epsilon_0}$ by Palmgren can be carried out as well in $\mathrm{ML}_1\mathrm{W}^-$.
  - $\mathrm{ML}_1\mathrm{W}^-$ can be interpreted in $\mathrm{ID}^{\mathrm{i}}_{<\epsilon_0}$
  - Therefore $|\mathrm{ML}_1\mathrm{W}^-| = |\mathrm{ID}^{\mathrm{i}}_{<\epsilon_0}| = \psi_{\Omega_1}(\Omega_{\epsilon_0})$.

Conclusion

- Palmgren showed that $\mathrm{ID}^{\mathrm{i}}_{<\epsilon_0}$ can be interpreted in $\mathrm{ML}_1\mathrm{W}$.
- Therefore $\mathrm{ML}_1\mathrm{W}$ shows the consistency of $(\Pi^1_1 - \mathrm{CA})_0$ sufficient to carry out most real mathematical proofs.
- Therefore Palmgren's result gives an answer to a revised Hilbert's program.
- The result can be sharped to determine the precise strength of a weaker theory $\mathrm{ML}_1\mathrm{W}^-$.

(Optional Slide) Proof of Palmgren

- Strictly positive inductive definitions give rise to a monotone operator

$$\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$$

  where $\mathcal{P}(\mathbb{N}) = \mathbb{N} \to \mathrm{U}$.
- For a strictly positive inductive definition one can "collect" all the sets, $\forall$-quantifiers in its definition are ranging over.
- Now one defines a W-type which has as branching degree all those sets.
- If we iterate the operator Γ transfinitely over the W-type, one obtains the least fixed point of Γ which one can use to interpret an inductive definition.
- By "Gentzen's trick" one obtains transfinite induction up to $< \epsilon_0$ over types, and can use it to get iterated inductive definitions up to α for any $\alpha < \epsilon_0$.

(Optional Slide) Further Result by Palmgren

- Erik Palmgren showed as well in [4] that if one replaces the W-type in type theory by finitely iterated versions of Aczel's V
  - Used by Aczel to interpret constructive set theory CZF in type theory

  one obtains the strength $|\mathrm{ID}^{\mathrm{i}}_{<\omega}| = \psi_{\Omega_1}(\Omega_\omega) = |(\Pi^1_1 - \mathrm{CA})_0|$
  (as noted before)

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