Assigning junior doctors to hospitals - what makes it so hard?

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Joint work with Georgios Askalidis, Péter Biró, Maxence Delorme, Tamás Fleiner, Sergio García, Jacek Gondzio, Nicole Immorlica, Rob Irving, Jörg Kalcsics, Augustine Kwanashie, Iain McBride, Eric McDermid, Shubham Mittal, William Pettersson Emmanouil Pountourakis and James Trimble
Centralised matching schemes

- Intending junior doctors must undergo training in hospitals
- Doctors rank hospitals in order of preference
- Hospitals do likewise with their applicants
- Centralised matching schemes (clearinghouses) produce a matching in several countries
  - US (National Resident Matching Program)
  - Canada (Canadian Resident Matching Service)
  - Japan (Japan Residency Matching Program)
  - Scotland (Scottish Foundation Allocation Scheme)
    - typically 700-750 applicants and 50 hospitals
- Stability is the key property of a matching
  - [Roth, 1984]
• Hospitals / Residents problem – classical results
• Size versus stability
• Ties
• Couples
• Lower quotas
• Social stability
• IP models
Hospitals / Residents problem (HR)

• Classical stable matching problem: the Hospitals / Residents problem (HR)

• We have $n_1$ doctors $d_1, d_2, \ldots, d_{n_1}$ and $n_2$ hospitals $h_1, h_2, \ldots, h_{n_2}$

• Each hospital has a capacity

• Doctors rank hospitals in order of preference, hospitals do likewise

• $d$ finds $h$ acceptable if $h$ is on $d$’s preference list, and unacceptable otherwise (and vice versa)

• A matching $M$ is a set of doctor-hospital pairs such that:
  1. $(d,h) \in M \Rightarrow d, h$ find each other acceptable
  2. No doctor appears in more than one pair
  3. No hospital appears in more pairs than its capacity
Each hospital has capacity 2

Doctor preferences

Hospitl preferences
Each hospital has capacity 2

Doctor preferences

\[ M = \{(d_1, h_1), (d_2, h_2), (d_3, h_3), (d_5, h_2), (d_6, h_1)\} \text{ (size 5)} \]
Matching $M$ is \textit{stable} if $M$ admits no \textit{blocking pair}

$$(d,h)$$ is a blocking pair of matching $M$ if:

1. $d$, $h$ find each other acceptable
   and
2. either $d$ is unmatched in $M$
   or $d$ prefers $h$ to his/her assigned hospital in $M$
   and
3. either $h$ is undersubscribed in $M$
   or $h$ prefers $d$ to its worst doctor assigned in $M$
HR: blocking pair (1)

Each hospital has capacity 2

Doctor preferences

\[ M = \{(d_1, h_1), (d_2, h_2), (d_3, h_3), (d_5, h_2), (d_6, h_1)\} \text{ (size 5)} \]

(d_2, h_1) is a blocking pair of \( M \)
Each hospital has capacity 2

\[
M = \{(d_1, h_1), (d_2, h_2), (d_3, h_3), (d_5, h_2), (d_6, h_1)\} \quad \text{(size 5)}
\]

\((d_4, h_2)\) is a blocking pair of \(M\)
Each hospital has capacity $2$

Doctor preferences

$$M = \{(d_1, h_1), (d_2, h_2), (d_3, h_3), (d_5, h_2), (d_6, h_1)\} \text{ (size 5)}$$

$(d_4, h_3)$ is a blocking pair of $M$
Each hospital has capacity 2

Doctor preferences

\[ M = \{(d_1, h_2), (d_2, h_1), (d_3, h_1), (d_4, h_3), (d_6, h_2)\} \] (size 5)

- \(d_5\) is unmatched
- \(h_3\) is undersubscribed
• A stable matching always exists and can be found in linear time [Gale and Shapley, ’62; Gusfield and Irving, ’89]

• There are doctor-optimal and hospital-optimal stable matchings

• Stable matchings form a distributive lattice [Conway, ’76; Gusfield and Irving, ’89]

• “Rural Hospitals Theorem”: for a given instance of HR:
  1. the same doctors are assigned in all stable matchings;
  2. each hospital is assigned the same number of doctors in all stable matchings;
  3. any hospital that is undersubscribed in one stable matching is assigned exactly the same set of doctors in all stable matchings.

  [Roth, ’84; Gale and Sotomayor, ’85; Roth, ’86]
A special case of HR arises when $n_1=n_2$, every hospital has capacity 1, and every doctor finds every hospital acceptable

- **Stable Marriage problem (SM)** [Gale and Shapley, ’62; Gusfield and Irving, ’89]

Also the case where $n_1=n_2$, every hospital has capacity 1, and not every doctor necessarily finds every hospital acceptable

- **Stable Marriage problem with Incomplete lists (SMI)** [Gale and Shapley, ’62; Gusfield and Irving, ’89]

In both cases the doctors and hospitals are more commonly referred to as the *men* and *women*
The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012
Alvin E. Roth, Lloyd S. Shapley

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012 was awarded jointly to Alvin E. Roth and Lloyd S. Shapley "for the theory of stable allocations and the practice of market design"
Hard variants of HR

• Hospitals / Residents problem – classical results
• Size versus stability
• Ties
• Couples
• Lower quotas
• Social stability
• IP models
Each hospital has capacity 1

<table>
<thead>
<tr>
<th>d₁: h₁ h₂</th>
<th>h₁: d₁ d₂</th>
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<tbody>
<tr>
<td>d₂: h₁</td>
<td>h₂: d₁</td>
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</table>
Maximum matchings versus stable matchings

Each hospital has capacity 1

Stable matching has size 1

Maximum matching has size 2
Maximum matchings versus stable matchings

Each hospital has capacity 1

Stable matching has size 1

Maximum matching has size 2

- Instance may be replicated to give arbitrarily large instances for which size of maximum matching is twice size of stable matching

- **Idea:** trade off size against stability, allowing larger matchings whilst tolerating a small amount of instability
Maximum matchings versus stable matchings

Each hospital has capacity 1

$M_1$ is stable

Blocking pairs of $M_2$: 

- $(d_3, h_2)$, 
- $(d_4, h_1)$

Blocking pair of $M_3$: 

- $(d_3, h_2)$

Must be optimal
Let $I$ be an HR instance

Given a matching $M$, let $bp(M)$ denote the set of blocking pairs relative to $M$ in $I$

Define $bp(I) = \min\{|bp(M)| : M \text{ is a maximum matching in } I\}$

A maximum matching $M$ in $I$ such that $|bp(M)| = bp(I)$ is called a maximum almost-stable matching

In an SMI instance, finding a maximum almost-stable matching is:
- NP-hard even if each preference list is of length $\leq 3$
- not approximable within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless P=NP
- polynomial-time solvable if doctors’ preference lists are of length $\leq 2$
- [Biró, M and Mittal, 2010]
- Open problem: HR where preference lists on one side are of length $\leq 2$
Hospitals / Residents problem with Ties

- In practice, doctors’ preference lists are short

- Hospitals’ lists are generally long, so ties may be used – *Hospitals / Residents problem with Ties (HRT)*

- A hospital may be *indifferent* among several doctors

- E.g., $h_1: (d_1, d_3) \rightarrow d_2 (d_5, d_6, d_8)$

- Matching $M$ is *stable* if there is no pair $(d, h)$ such that:
  1. $d, h$ find each other acceptable
  2. *either* $d$ is unmatched in $M$
     - or $d$ prefers $h$ to his/her assigned hospital in $M$
  3. *either* $h$ is undersubscribed in $M$
     - or $h$ prefers $d$ to its worst doctor assigned in $M$
Each hospital has capacity 2

Doctor preferences

Hospital preferences

\(d_1: h_1 \ h_2\)
\(d_2: h_1 \ h_2\)
\(d_3: h_1 \ h_3\)
\(d_4: h_2 \ h_3\)
\(d_5: h_2 \ h_1\)
\(d_6: h_1 \ h_2\)

\(h_1: d_1 \ d_2 \ d_3 \ d_5 \ d_6\)
\(h_2: d_2 \ d_1 \ d_6 \ (d_4 \ d_5)\)
\(h_3: d_4 \ d_3\)
Each hospital has capacity 2

Doctor preferences

Hospital preferences

\[ M = \{(d_1, h_1), (d_2, h_1), (d_3, h_3), (d_4, h_2), (d_6, h_2)\} \text{ (size 5)} \]
HRT: stable matching (2)

Each hospital has capacity 2

Doctor preferences

\[ M = \{(d_1, h_1), (d_2, h_1), (d_3, h_3), (d_4, h_3), (d_5, h_2), (d_6, h_2)\} \] (size 6)

Hospital preferences

\[ M = \{(d_1, h_1), (d_2, h_1), (d_3, h_3), (d_4, h_3), (d_5, h_2), (d_6, h_2)\} \] (size 6)
Maximum size stable matchings

- Stable matchings can have different sizes
- A maximum stable matching can be (at most) twice the size of a minimum stable matching
- Problem of finding a maximum stable matching (MAX HRT) is NP-hard [Iwama, M et al, 1999], even if (simultaneously):
  - each hospital has capacity 1 (Stable Marriage problem with Ties and Incomplete Lists)
  - each doctor’s preference list is strictly ordered and of length $\leq 3$
  - each hospital’s preference list is either:
    • strictly ordered and of length $\leq 3$
    • a tie of length 2
    [McDermid and M, 2010]

- Minimisation problem is NP-hard too, for similar restrictions! [M et al, 2002]
MAX HRT: approximability

- **Upper bounds:**
  - trivial $2$-approximation algorithm for MAX HRT
  - succession of papers gave improvements, culminating in:
    - MAX HRT is approximable within $3/2$ [McDermid, 2009; Király, 2012; Paluch 2012]
    - MAX HRT is approximable within $(1+1/e) \approx 1.3679$ for ties on one side only [Lam and Plaxton, 2019]

- **Lower bounds:**
  - MAX HRT is not approximable within $33/29$ unless P=NP, even if each hospital has capacity $1$ [Yanagisawa, 2007]
  - MAX HRT is not approximable within $4/3-\varepsilon$ assuming the *Unique Games Conjecture* (UGC) [Yanagisawa, 2007]

- **Open problems:**
  - increase lower bounds / decrease upper bounds
Pairs of doctors who wish to be matched to geographically close hospitals form *couples*

Each couple \((d_i,d_j)\) ranks in order of preference a set of pairs of hospitals \((h_p,h_q)\) representing the assignment of \(d_i\) to \(h_p\) and \(d_j\) to \(h_q\)

Hospitals rank individual doctors as before

Stability definition may be extended to this case [Roth, 1984; McDermid and M, 2010; Biró et al, 2011]

Gives the *Hospitals / Residents problem with Couples* (HRC)

A stable matching need not exist

Stable matchings can have different sizes
The problem of determining whether a stable matching exists in a given HRC instance is

- NP-complete, even if each hospital has capacity 1 and:
  - there are no single doctors
    [Ng and Hirschberg, 1988; Ronn, 1990]
  - there are no single doctors, and
  - each couple has a preference list of length ≤2, and
  - each hospital has a preference list of length ≤2
    [Biró, M and McBride, 2014]

- solvable in polynomial time if:
  - each single doctor has a preference list of length ≤2, and
  - each couple has a preference list of length 1, and
  - each hospital has a preference list of length ≤2
    [M, McBride and Trimble, 2016]

- **Open problem**: resolve complexity for other restricted cases
• In the *Hospitals / Residents problem with Lower Quotas* (HR-LQ), each hospital has a *lower quota* as well as its upper quota (capacity).

• In a matching $M$ each hospital $h_j$ must satisfy $|M(h_j)|=0$ ($h_j$ is *closed*) or $l_j \leq |M(h_j)| \leq c_j$ where $l_j$ and $c_j$ are the lower and upper quotas.

• $M$ is *stable* if it admits no blocking pair and no *blocking coalition*.
  
  o A *blocking coalition* of $M$ involves a closed hospital $h_j$ and a set of $l_j$ doctors, each of whom is unmatched or prefers $h_j$ to his/her assigned hospital in $M$.

• An instance of HR-LQ need not admit a stable matching.

Doctors
\begin{align*}
d_1 & : h_1 \ h_2 \\
d_2 & : h_2 \ h_1
\end{align*}

Hospitals
\begin{align*}
h_1 & : 2 : 2 : d_1 \ d_2 \\
h_2 & : 1 : 1 : d_1 \ d_2
\end{align*}
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<td>$h_1 : 2 : 2 : d_1, d_2$</td>
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Hospitals

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$M$ is *stable* if it admits no blocking pair and no blocking coalition.

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An instance of HR-LQ need not admit a stable matching.

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</tr>
<tr>
<td>$h_1$: $d_2$</td>
<td>$d_1$:</td>
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The problem of deciding whether an instance of HR-LQ admits a stable matching is NP-complete even if each upper quota $\leq 3$.

[Biró, Fleiner, Irving and M, 2010]

Open problem: complexity for lower / upper quotas $\leq 2$. 

• Although pairs may block a matching $M$ in theory, there is no guarantee they will block $M$ in practice

• If no social ties exist between pairs they are far less likely to form blocking pairs
  o if they do not know about each other’s preferences and matched partners

• Relaxing the stability definition to consider only pairs that are likely to block a matching in practice gives the Hospitals / Residents problem under Social Stability (HRSS)

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<td>$h_3$: $d_4$ $d_3$</td>
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<td>$d_4$: $h_2$ $h_3$</td>
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<tr>
<td>$d_5$: $h_2$ $h_1$</td>
<td></td>
</tr>
<tr>
<td>$d_6$: $h_1$ $h_2$</td>
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</table>

Each hospital has capacity 2

Unacquainted pairs $U=\{(d_1, h_2), (d_3, h_1), (d_5, h_2)\}$
An instance \((I, G)\) of HRSS consists of:

- An HR instance \(I\)
- A social network graph \(G = (D \cup H, A)\)
- Edges in \(G\) are called \textit{acquainted} pairs

Unacquainted pairs \(U = \{(d_1, h_2), (d_3, h_1), (d_5, h_2)\}\)

Each hospital has capacity 2

**Social stability definition in the HRSS context** – \textit{social stability}
• A pair \((d, h)\) forms a **social blocking pair** with respect to \(M\) if
  
  o \((d, h)\) blocks \(M\) in the classical sense
  
  o \((d, h)\) is an acquainted pair

• A **socially stable matching** is one that admits no social blocking pairs

• In practice the social network graph may be inferred on the basis of agents’ previous interactions with one another

• Agents do not need to be acquainted in order to find one another acceptable

• Given HR and HRSS instances \(I\) and \((I, G)\) respectively, any stable matching in \(I\) is also socially stable in \((I, G)\)
Each hospital has capacity 2

Unacquainted pairs $U = \{(d_1, h_2), (d_3, h_1), (d_5, h_2)\}$

Socially Stable Matching $M = \{(d_1, h_2), (d_2, h_1), (d_3, h_1), (d_4, h_3), (d_6, h_2)\}$

$|M| = 5$
Each hospital has capacity 2

Unacquainted pairs \( U = \{(d_1, h_2), (d_3, h_1), (d_5, h_2)\} \)

Socially Stable Matching \( M' = \{(d_1, h_2), (d_2, h_1), (d_3, h_3), (d_4, h_3), (d_5, h_1), (d_6, h_2)\} \)

\( |M'| = 6 \)

- An instance of HRSS can admit socially stable matchings of varying sizes
- Socially stable matchings can be larger than stable matchings
  - can be twice the size of stable matchings in a given instance
The following complexity results are known:

- NP-complete to determine if an instance of the Stable Roommates problem with Free Pairs (variant of HRSS for one set of agents) admits a socially stable matching [Cechlárová and Fleiner, ’09]

- Finding a maximum size socially stable matching in HRSS is:
  - NP-hard, even if each hospital has capacity 1 and each preference list is of length \( \leq 3 \)
  - solvable in polynomial time if each hospital has capacity 1 and each preference list on one side is of length \( \leq 2 \)
  - solvable in polynomial time if either \(|U|=k\) or \(|A|=k\) for some constant \(k\)
  - approximable within a factor of \(3/2\)
  - not approximable within \(3/2-\epsilon\) for any \(\epsilon>0\) assuming UGC

[Askalidis, Immorlica, Kwanashie, M and Pountourakis, 2013]

- Open problems: complexity in the presence of:
  - master lists
  - ties
Integer Programming model for MAX HRT

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{n_1} \sum_{h_j \in P(d_i)} x_{i,j} \\
\text{subject to} & \quad \sum_{h_j \in P(d_i)} x_{i,j} \leq 1 \quad (1 \leq i \leq n_1) \\
& \quad \sum_{d_i \in P(h_j)} x_{i,j} \leq c_j \quad (1 \leq j \leq n_2) \\
& \quad c_j \left(1 - \sum_{h_q \in S_{i,j}} x_{i,q}\right) - \sum_{d_p \in T_{i,j}} x_{p,j} \leq 0 \quad (1 \leq i \leq n_1, h_j \in P(d_i)) \\
x_{i,j} & \in \{0, 1\}
\end{align*}
\]
Scottish Foundation Allocation Scheme

- Ran from 1999-2012

- Each doctor:
  - ranked up to 10 hospitals in strict order of preference
  - had an integral score in the range 40..100

- Each hospital:
  - had a capacity indicating its number of posts
  - had a preference list derived from the above scoring function
  - so ties were possible
With basic model [Kwanashie and M, 2014]

| Year | Doctors | Hospitals | Posts | $|M|$ | Time (sec) |
|------|---------|-----------|-------|-----|-----------|
| 2008 | 748     | 52        | 752   | 709 | 75.5      |
| 2007 | 781     | 53        | 789   | 746 | 21.8      |
| 2006 | 759     | 53        | 801   | 758 | 93.0      |

More sophisticated model:
- dummy variables
- constraint merging
- preprocessing and warm start
- SFAS instances solved in 5 seconds on average
- [Delorme et al, 2019]
Conclusions

- Classical HR problem has nice structure and algorithms
- Many variants with practical applications are NP-hard:
  - maximum almost-stable matchings
  - MAX HRT
  - HRC
  - HR-LQ
  - HRSS
- Integer Programming can be used to find optimal solutions in some cases
- Future work:
  - find boundaries between P and NP-hard cases
  - approximation algorithms
  - FPT algorithms
  - scale up IP models to work with larger instance sizes
Grants EP/P028306/1 and EP/P029825/1

Grants EP/K010042/1 and EP/K01000X/1

Grant EP/E011993/1