Parameterized Pre-Coloring Extension and List Coloring Problems

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Outline

Parameterized Problem

- A *parameterized problem* is a language $L \subseteq \Sigma^* \times \mathbb{N}$. Input instance of L is (x, k) where $x \in \Sigma^*, k \in \mathbb{N}$. k is called *parameter*.

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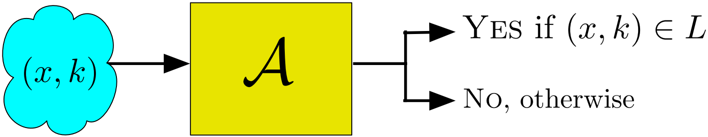
- Example: Vertex Cover parameterized by Solution Size. $L = \{(G, k) | \exists S \subseteq V(G)$ such that $|S| \leq k$ and $G \setminus S$ has no edge$\}$.

Fixed-Parameter Tractability (FPT)



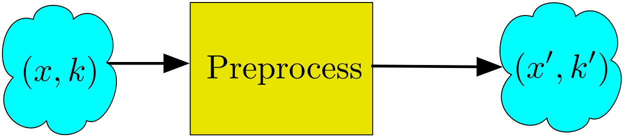
- Algorithm $\mathcal{A}$ runs in $f(k) \cdot |x|^c$ time.
- $\mathcal{A}$ is called FIXED PARAMETER ALGORITHM.

Hardness in Parameterized Complexity

Hardness in Parameterized Complexity

$$\text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \ldots \subseteq \text{XP}.$$

Kernelization



- Preprocessing takes $poly(|x|, k)$ time.
- $(x, k) \in L$ if and only if $(x', k') \in L$.
- $|x'| + k' \leq g(k)$.
- If $g(k) = poly(k)$, then we say that L has a polynomial kernel.

Graph Coloring

Graph Coloring

p-COLORING
Input: An undirected graph $G = (V, E)$ and a set of p colors Q.
Goal: Does there exist $\lambda : V(G) \to Q$ such that for every $u, v \in V(G)$, $\lambda(u) \neq \lambda(v)$?

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- For $p \geq 3$, p-COLORING is NP-Complete in general graphs.

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- For $p \geq 3$, p-Coloring is NP-Complete in general graphs.
- p-Coloring is polynomial time solvable on chordal graphs.

Precoloring Extension

Pre-coloring Extension
Input: A graph G, and a precoloring $\lambda_P : X \to Q$ for $X \subseteq V(G)$ where Q is a set of colors.
Goal: Can λ_P be extended to a proper coloring of G using colors from only Q?

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- PRE-COLORING EXTENSION is polynomial time solvable in cluster graphs, but NP-Complete in bipartite graphs.

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- LIST COLORING is NP-Complete in split graphs, and graphs of cliquewidth two.

Outline

Our problems and results

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Pre-Coloring Extension Clique Modulator
Input: A graph G, a clique modulator D with at most k vertices, and a precoloring $\lambda_P : X \mapsto Q$ for $X \subseteq V(G)$ where Q is a set of colors.
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What is known

From Paulusma (WG 2015)

Parameter	**Coloring**	**Pre-Color Ext**	**List-Color**
clique-width	W[1]-hard	para-NPC	para-NPC
treewidth	FPT	W[1]-hard	W[1]-hard
cluster deletion	FPT	W[1]-hard	W[1]-hard
vertex cover	FPT	FPT	W[1]-hard

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- Golovach, Paulusma, and Song (2014) asked to determine the parameterized complexity status of PRE-COLORING EXTENSION CLIQUE MODULATOR.
- We prove positively that PRE-COLORING EXTENSION CLIQUE MODULATOR is FPT.

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- We prove that Pre-Coloring Extension Clique Modulator admits a kernel with $3k$ vertices.

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Input: A graph G, a list $L(v)$ of $(n-k)$ colors for every $v \in V(G)$.
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- An instance (G, L, k) of $(n-k)$-REGULAR LIST COLORING can be transformed into an equivalent instance (G', D, L, k') of $(n-k)$-REGULAR LIST COLORING such that $k' = 2k$.

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- We prove that List Coloring Clique Modulator admits a randomized algorithm running in time $\mathcal{O}^*(2^k)$.
- We prove that List Coloring Clique Modulator admits no polynomial kernel unless NP $\subseteq$ coNP/poly.

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- We answer positively by proving that $(n-k)$-REGULAR LIST COLORING admits a polynomial kernel with $\mathcal{O}(k^2)$ vertices and colors.
- We also provide a compression to a variation of the problem with $11k$ vertices and $\mathcal{O}(k^2)$ colors, encodable in $\mathcal{O}(k^2 \log_2 k)$ bits.

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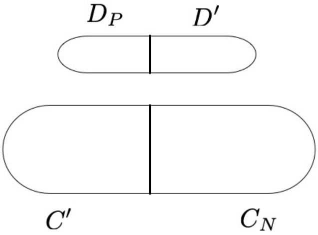

D_P: precolored vertices from D.

$C_N = \{v \in C | \exists u \in D', uv \notin E(G)\}$

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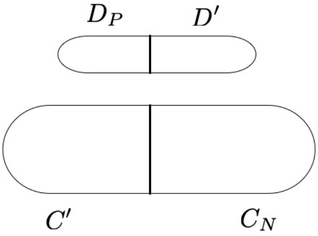

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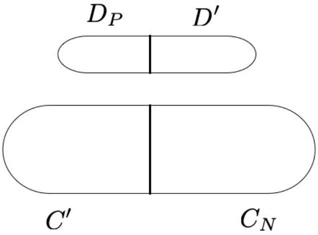

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- When Rule 1 is not applicable, and $|Q| \geq |C|$, hence $|C_N| \leq k^2$.

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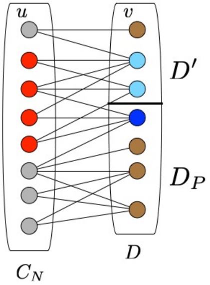

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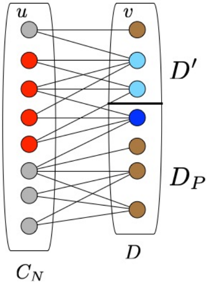

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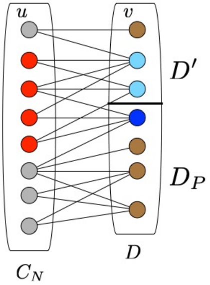

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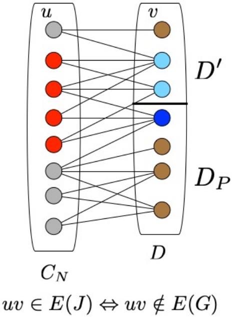

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- If Rule 2 is not applicable, then $|C_N| \leq |D| \leq k$.

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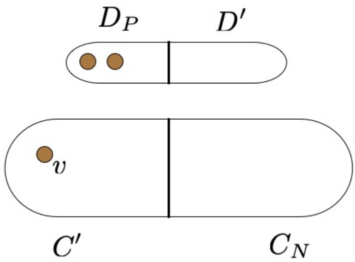

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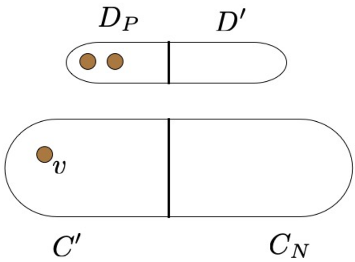

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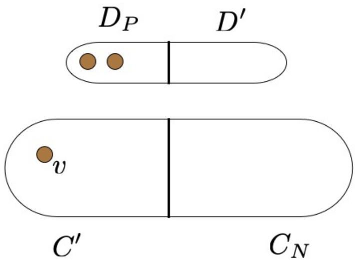

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- When Rule 3 is not applicable, there is no pre-colored vertex in C'.

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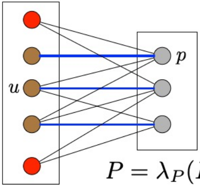
$\exists v \in V(G)$ s.t. $(u, v) \notin E(G)$ and $\lambda_P(v) \neq p$

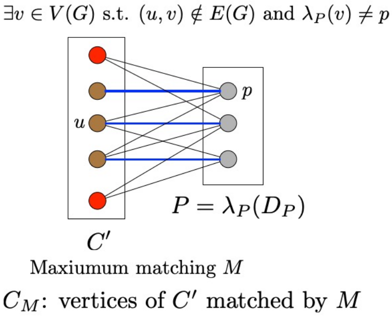


$P = \lambda_P(D_P)$

C'

Maxiumum matching M

C_M: vertices of C' matched by M



- Construct bipartite graph $H = (C, P)$ as above.

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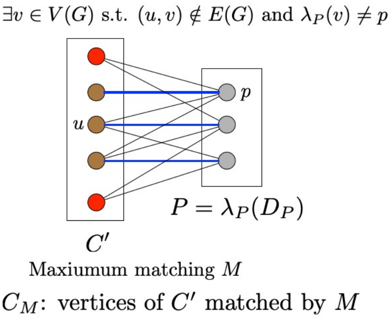

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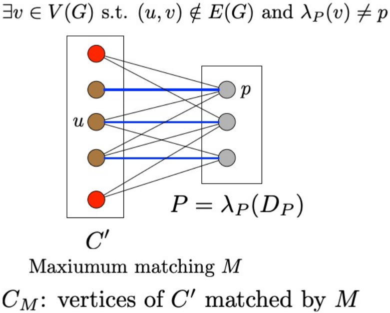

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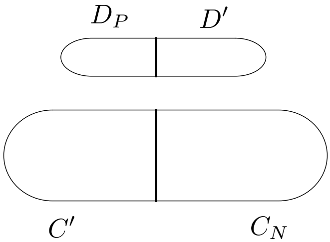
- If Rule 4 is not applicable, then $|C'| \leq |P| \leq |D_P| \leq k$.

Precoloring Extension: Result



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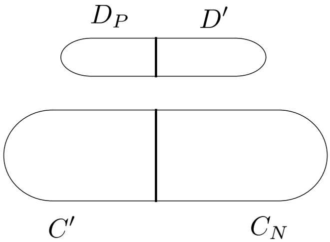

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- $|C_N| \leq |D| \leq k, |C'| \leq |D_P| \leq k$.

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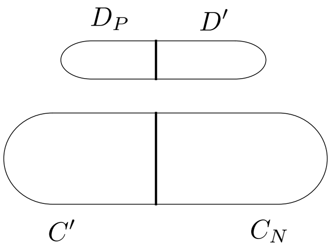

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- $|C_N| \leq |D| \leq k, |C'| \leq |D_P| \leq k$.
- Hence, Pre-Coloring Extension Clique Modulator has a kernel with $3k$ vertices.

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- For $(n-k)$-Regular List Coloring, can we get a kernel with $\mathcal{O}(k)$ veritces?

THANK YOU