Formal theorems of Intuitionistic type theory

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Outline

- The Modulus of Continuity
- Bar Induction
- What can we prove using them?

The Modulus of Continuity

- $\mathbb{S} = \mathbb{N} \to T$ where T is an inhabited subtype of $\mathbb{N}$.
- $\mathcal{F} = \mathbb{S} \to \mathbb{N}$ and $F \in \mathcal{F}$ and $f \in \mathbb{S}$, so $F(f) \in \mathbb{N}$.
- $F(f)$ depends on only a finite part of f. How much?
- $f_n^e(x) =$ if $x < n$ then $f(x)$ else exception(e, x)
- $M_F(n, f) = \nu e.\ F(f_n^e)?e : x.\langle x, x\rangle$
- $M_F(n, f) \in \mathbb{N} \cup (\mathbb{N} \times \mathbb{N})$
- (A) If $M_F(n, f) = k \in \mathbb{N}$ then $k = F(f)$ and
  $\forall m \geq n.\ M_F(m, f) = k$.
- (B) $\exists n{:}\mathbb{N}.\ M_F(n, f) = F(f)$.
- (Kleene M) $KM = \lambda F.\ M_F$
- $KM \in \mathcal{F} \to \{\mathbb{N} \to \mathbb{S} \to \mathbb{N} \cup (\mathbb{N} \times \mathbb{N}) \mid A \wedge B\}$?? No!
- $KM \in \mathcal{F} \to \{\mathbb{N} \to \mathbb{S} \to \mathbb{N} \cup (\mathbb{N} \times \mathbb{N}) \mid A \wedge B\}//\text{True}$

Bar Induction

- T a type, $Seq(T) = k{:}\mathbb{N} \times (\mathbb{N}_k \to T)$, nil $= \langle 0, \bot \rangle$.
- $B \in Seq(T) \to \mathbb{P}$ is a *bar* if $\forall f{:}\mathbb{N} \to T.\ \exists n{:}\mathbb{N}.\ B(\langle n, f \rangle)$.
- B is *decidable* if $\forall s{:}Seq(T).\ (B(s) \vee \neg B(s))$
- $Q \in Seq(T) \to \mathbb{P}$ is a *inductive* if
  $$\forall s{:}Seq(T).\ (\forall t{:}T.\ Q(s + [t])) \Rightarrow Q(s).$$
- $(B \Rightarrow Q)$ if $\forall s{:}Seq(T).\ B(s) \Rightarrow Q(s)$
- Bar Induction: If B is a decidable bar and Q is inductive and $(B \Rightarrow Q)$ then $Q(\text{nil})$.
- Remarks:
  - Bar Induction is true classically.
  - Bar Induction is false if all $f \in \mathbb{N} \to \mathbb{N}$ are recursive.
  - We assume Bar Induction only for $Q(s)$ of the form $F(s) \in X(s)$ that have no constructive content.
  - We use that to prove that *bar recursion* realizes Bar Induction for general Q.

Why use Intuitionistic Type Theory?

- We can prove stronger theorems of constructive analysis than are provable using only Bishop's constructive analysis.
  - Strong connectedness of the continuum.
  - Brouwer's uniform continuity theorem.
  - Strong form of Brouwer's (approximate) fixedpoint theorem.
  - Simple definition of derivative and better Chain Rule.
- We can derive useful induction principles from Bar Induction.
  - Transfinite Induction (on well-founded relations)
  - Induction on W-types and parameterized families of W-types. (Similar to Coq's *inductive construction*)
- Using continuity, we derive induction on *monotone* bars.
  - B is *monotone* if $\forall s{:}Seq(T).\ (B(s) \Rightarrow \forall t{:}T.\ B(s + [t]))$
  - We can do Bezem & Veldman's original proof of the intuitionistic Ramsey's theorem (intersections of almost full relations is almost full).

Strong Connectedness of the Continuum

- The reals $\mathbb{R}$ are convergent (regular) sequences of rationals. There is an equivalence relation $x \equiv y$, but we do not form the quotient type. (We use the *setoid* $\mathbb{R}, \equiv$).
- A *set* of reals is a proposition $P(x)$ such that for all $x, y \in \mathbb{R}$, $(P(x) \ \wedge \ x \equiv y) \Rightarrow P(y)$
- Using continuity we proved: If A and B are inhabited sets of reals and $\mathbb{R} \subseteq (A \cup B)$ then $(A \cap B)$ is inhabited.
- Remarks:
  - In classical math, A and B need to be *open* sets.
  - Brouwer proved $\neg(A \cap B = \emptyset)$
  - Mike Shulman, trying to connect homotopy type theory and constructive analysis, introduced a concept of *crisp* sets of reals. One of his axioms is that two crisp sets that cover the reals have a point in common. We discovered our theorem while trying to interpret "crisp" in Nuprl.
  - We conjecture that connecting HoTT and constructive analysis will work best using intuitionistic math (viz. with continuity and FAN).

Brouwer's uniform continuity theorem

- X and Y are (pseudo)metric spaces and $f \in X \to Y$.
- f is a *metric operation* if for $x_1, x_2 \in X$,
  $$d(x_1, x_2) = 0 \Rightarrow d(f(x_1), f(x_2)) = 0.$$
- f is *uniformly continuous* if $\forall \epsilon > 0.\ \exists \delta > 0.$
  $\forall x_1, x_2 \in X.\ d(x_1, x_2) < \delta \Rightarrow d(f(x), f(y)) < \epsilon.$
- *compact* = *complete* and *totally bounded*. X is complete if every Cauchy sequence in X converges in X. X is totally bounded if for every $\epsilon > 0$ there is a finite list L of points in X such that every point in X is within ϵ of a point in L.

UCT If X is compact then f is uniformly continuous if and only if f is a metric operation. Proof uses FAN + CONT

- Remarks:
  - We use terminology *metric operation* rather than *function* to avoid contradicting classical math.
  - Metric operations are closed under composition. Bishop's analysis can not prove that uniformly continuous functions are closed under composition.

Brouwer's fixedpoint theorem

- $B(n)$ is the unit n-dimensional ball, $\{x \in \mathbb{R}^n \mid d(x, 0) \leq 1\}$. It is *compact*.
- Theorem: For every metric operation f from $B(n)$ to $B(n)$ and every $\epsilon > 0$ there exists $x \in B(n)$ with $d(x, f(x)) < \epsilon$.
  - In BISH one must also assume f is uniformly continuous.
  - In CLASS one must assume f is pointwise continuous, but get the "existence" of an exact fixedpoint $d(x, f(x)) = 0$.
- We adapted an inductive proof by Karol Sieclucki that there is no retraction from $|K| \to |\partial K|$ for an n-dimensional *rational cubical complex* K.
- The existence of approximate fixedpoints follows from this no-retraction theorem.

Better definition of derivative.

- I is an interval and $f, f' \in I \to \mathbb{R}$.
- $df(x)/dx = f'$ if
  $\forall x \in I. \lim_{y \to x, y \in I}(f(y) - f(x)/y - x) = f'(x)$.
- Bishop's definition is: for every compact sub-interval $J \subseteq I$, for every $\epsilon > 0$ there exists $\delta > 0$ such that $\forall x, y \in J$.
  $|x - y| < \delta \Rightarrow |(f(y) - f(x) - f'(x)(y - x)| < \epsilon * |y - x|$.
- Using UCT we can prove these two equivalent.
- (Chain rule) If $f, f' \in I \to J$ and $g, g' \in J \to \mathbb{R}$ and $df(x)/dx = f'$ and $dg(x)/dx = g'$ then
  $d(g(f(x))/dx = g'(f(x)) * f'(x)$.
- Bishop must assume in addition that f maps each compact subinterval of I into a compact subinterval of J. This is often troublesome to prove.