# Properties of Boolean Algebras

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# Plan For Today

- Boolean Algebras and Order.
- Subalgebras.
- Homomorphisms.
- Ideals and Filters.
- The Homomorphism Theorem.

# Brief Re-cap

**Definition**

A Boolean algebra is a set $A$ together with

- the distinguished elements 0 and 1,
- the binary operations $\wedge$ (meet) and $\vee$ (join),
- and the unary operation $'$ (complement).

Such that certain axioms hold.

# Boolean Algebra From CASL Library

**spec** BOOLEANALGEBRA =
  **sort** *Elem*
  **ops** 0, 1 : *Elem*;
    $\_\_\wedge\_\_$ : *Elem* × *Elem* → *Elem*, *assoc*, *comm*, *unit* 1;
    $\_\_\vee\_\_$ : *Elem* × *Elem* → *Elem*, *assoc*, *comm*, *unit* 0
  $\forall\ x, y, z : Elem$
  • $x \wedge (x \vee y) = x$ %(absorption_def1)%
  • $x \vee x \wedge y = x$ %(absorption_def2)%
  • $x \wedge 0 = 0$ %(zeroAndCap)%
  • $x \vee 1 = 1$ %(oneAndCup)%
  • $x \wedge (y \vee z) = x \wedge y \vee x \wedge z$ %(distr1)%
  • $x \vee y \wedge z = (x \vee y) \wedge (x \vee z)$ %(distr2)%
  • $\exists\ x' : Elem \bullet x \vee x' = 1 \wedge x \wedge x' = 0$ %(inverse)%

**end**

A simple example of a Boolean algebra, is given a set $A = \{2, 4, 6\}$, we then define the Boolean algebra: $B = (\mathcal{P}(A), \wedge, \vee, ', 1 = A, 0 = \emptyset)$.

This is an example we shall use throughout, to give a feel for the notions we shall present.

# Order

# Defining Order

Given a set $A$, in $\mathcal{P}(A)$ the notion of set inclusion can be defined by $\cap$ or $\cup$:

- $p \subseteq q \iff p \cap q = p$, and equivalently
- $p \subseteq q \iff p \cup q = q$

**Definition**

$p \leq q$ in the case that $p \wedge q = p$.

Note: The relation $\leq$ is a partial order, i.e.,

- $p \leq p$, reflexive,
- if $p \leq q$ and $q \leq p$ then $p = q$, antisymmetric, and
- if $p \leq q$ and $q \leq r$ then $p \leq r$, transitive.

These all follow from the idempotency, commutativity and associativity of $\wedge$ and $\vee$.

**Proof In Hets.**

# Justification of the Definition of Order

**Lemma**

*$p \wedge q = p$ if and only if $p \vee q = q$*

**Proof.**

Since $p \wedge q = p$ then $p \vee q = (p \wedge q) \vee q$ and now it follows from the axiom of absorption $(p \wedge q) \vee q = q$. □

**Corollary**

*Defining $\leq$ in terms of $\wedge$ is equivalent to defining $\leq$ in terms of $\vee$.*

# Implications of Order

Given this ordering, we can now reveal some interesting facts about Boolean algebras, namely

- If $p \leq q$ and $r \leq s$ then both $p \wedge r \leq q \wedge s$ and $p \vee r \leq q \vee s$.
- If $p \leq q$, then $q' \leq p'$.
- $p \leq q$ if and only if $p - q = 0$.

The proofs of these assertions follow directly from the definition of $\leq$ and the axioms of Boolean algebras.

# Definition of Supremum

**Definition**

Let $(P, \leq)$ be a partial order, Let $X \subseteq P$ be a set.
$u \in P$ is called an upper bound of $X$ if $\forall x \in X.\ x \leq u$.
A element $s$ is a supremum of $X$, if it is an upper bound of $X$ and $\forall s'.\ s'$ is an upper bound $\Rightarrow s \leq s'$

# Definition of Infimum

## Definition

Let $(P, \leq)$ be a partial order, Let $X \subseteq P$ be a set.
$u \in P$ is called an lower bound of $X$ if $\forall x \in X.\ u \leq x$.
A element $s$ is an infimum of $X$, if it is a lower bound of $X$ and $\forall s'.\ s'$ is an lower bound $\Rightarrow s' \leq s$

### Theorem

*Any finite subset E of a Boolean algebra A has both a supremum and an infimum.*

### Proof.

- If E is empty then every element is an upper and lower bound, therefore E has supremum 0 and infimum 1.
- If E is a singleton set, e.g. $\{s\}$, then $s$ is both the supremum and infimum.
- Finally consider sets with two elements. They have supremum $p \vee q$ and infimum $p \wedge q$.

□

**Proof on whiteboard.**

Remark:
Let FinAndCofin $= \{X \subseteq \mathbb{N} \,|\, X \text{ finite } \vee \ \mathbb{N} - X \text{ finite}\}$ be a Boolean algebra.

Then $\mathrm{E} = \{\{n\} \mid n \text{ even}\}$ does not have a supremum in FinAndCofin.

# Subalgebras

# Boolean Subalgebras

## Definition

Let $A$ be a Boolean algebra and $B$ a non-empty subset of $A$, such that

- if $a \in B$, then $a' \in B$,
- if $a, b \in B$, then $a \vee b \in B$.

Then $B$ – together with the original operations on $A$ restricted to arguments in $B$ – is called a subalgebra of $A$.

Remark: Leaving out $\wedge$ will be justified later.

# Examples

- The trivial subalgebras: $B_{min} = \{0, 1\}$; $B_{max} = A$.
- If $A$ contains a non-trivial element $a \notin \{0, 1\}$, then $B = \{0, 1, a, a'\}$ for $a \in A$ is a subalgebra.
- Examples on $\mathcal{P}(\{2, 4, 6\})$.

# Justification of The Subalgebra Definition

**Lemma**

*Let $B$ be a subalgebra of $A$ then,*

$$\text{if } a, b \in B, \text{ then } a \wedge b \in B.$$

**Proof.**

By De-Morgan laws. □

Remark: Alternatively, the definition could use $\wedge$, rather than $\vee$, followed by a lemma on closedness under $\vee$.

# Observations on Suprema and Infima

The definition of a subalgebra does not mention the notions of supremum and infimum, for this reason anything can happen with regards to them.

Observation 1:
With a finite Boolean algebra suprema and infima are preserved.
Observation 2:
With our FinAndCofin example the supremum of E is in $\mathcal{P}(\mathbb{N})$, but $E$ has no supremum in FinAndCofin.

# Homomorphisms

# Boolean Algebra Homomorphisms

## Definition

A Boolean Homomorphism is a mapping $f$ from a Boolean algebra $B$ to a Boolean algebra $A$ such that,

1. $f(p \wedge q) = f(p) \wedge f(q)$.
2. $f(p \vee q) = f(p) \vee f(q)$.
3. $f(p') = (f(p))'$

for all $p$ and $q$ in $B$.

Remark: The consequences for 0 and 1 will be discussed later.

# Special Homomorphisms

- Monomorphism - A one-to-one homomorphism.
- Epimorphism - A onto homomorphism.
- Isomorphism - A one-to-one and onto homomorphism.
- Endomorphism - A homomorphism from a Boolean algebra to itself.
- Automorphisms - A one-to-one and onto endomorphism.

# Homomorphisms and 0 and 1

As with subalgebras, the distinguished elements have a special role. From the definition of a homomorphism we have that,

$$f(p \wedge p') = f(p) \wedge (f(p))'$$

Therefore we have that,

- $f(0) = 0$, and by the dual argument
- $f(1) = 1$.

This leads to the consequence that the trivial mapping that maps all elements to 0 is not a homomorphism.

# Further Properties of Homomorphisms

Since all Boolean operations can be defined from $\wedge$, $\vee$ and $'$, including the order relation, it follows that Boolean homomorphisms are order preserving.

If a homomorphism preserves all suprema, and consequently infima, then it is known as a complete homomorphism.

# Homomorphism Example

Consider a subalgebra $B$ of a Boolean algebra $A$. The embedding from $B$ into $A$ is a homomorphism.

The identity mapping of any Boolean algebra is an automorphism.

The mapping $h : (\{0,1\}, \vee, \wedge, ', 1, 0) \rightarrow (B, \vee, \wedge, ', 1_B, 0_B)$ where $h(0) = 0_B$, and $h(1) = 1_B$ is a homomorphism.

Example on $\mathcal{P}(A)$

# Ideals and Filters

# Kernels

## Definition

Given a homomorphism $f$ from a Boolean algebra $A$ to a Boolean algebra $B$, its kernel is defined as the set,

$$K = \{p : f(p) = 0\}, \text{ i.e. the elements that } f \text{ maps to } 0.$$

# Boolean Ideals

## Definition

A Boolean ideal of a Boolean algebra $A$, is a subset $I$ of $A$ such that,

- $0 \in I$.
- If $p \in I$ and $q \in I$ then $p \vee q \in I$.
- If $p \in I$ and $q \in A$ then $p \wedge q \in I$.

Remark: $A$ is an ideal.

Proof on whiteboard: Every kernel is an ideal. (And Example)

# Boolean Filters

The dual notion of an ideal, is that of a filter.

**Definition**

A Boolean filter of a Boolean algebra $A$, is a subset $F$ of $A$ such that,

- $1 \in F$.
- If $p \in F$ and $q \in F$ then $p \wedge q \in F$.
- If $p \in F$ and $q \in A$ then $p \vee q \in F$.

# Properties of Ideals

Given that a kernel is in fact an ideal, we can conclude that every homomorphism gives us an ideal, namely its kernel.

We can also define an ideal to be a proper ideal if and only if it does not contain 1, since if the ideal was to contain 1 then by definition, it would contain the full Boolean algebra. Such an ideal we shall call an in-proper ideal.

Finally we can again give the notion of a maximal ideal, which is a proper ideal that is not included in any other proper ideal.

# Maximal Ideals

Maximal Ideals have some interesting properties, including

**Lemma**

*An ideal $M$ in a Boolean algebra $B$ is maximal if and only if $p \in M$ or $p' \in M$, but not both, for each $p \in B$.*

This lemma leads us to the fact that every maximal ideal is the kernel of some homomorphism, namely $f$ where

$$f(p) = 0 \text{ or } 1 \text{ if } p \in M \text{ or } p' \in M$$

Here $f$ is clearly a homomorphism to 2 and the kernel of $f$ is clearly $M$.

# The Homomorphism Theorem

# The Homomorphism Theorem

The property that every maximal ideal is the kernel of some homomorphism is a specific instance of a more general theorem, namely,

**Theorem**

*Every proper ideal is the kernel of some epimorphism.*

This theorem is known as the homomorphism theorem.

# The Homomorphism Theorem – Proof

**Proof.**

Given a Boolean algebra $B$ and a propel ideal $M$ of $B$, Construct the Boolean algebra $B/M$. That is construct the quotient of $B$ modulo $M$.

To do this use the equivalence relation

$$a \backsim b \iff (a \wedge b') \vee (a' \wedge b) \in M.$$

Now the epimorphism we want is the mapping from $B$ into $B/M$. Thus the kernel of the epimorphism is $M$. □

# References

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